Quasi-Lie structure of σ-derivations of $\mathbb{C}[t^{\pm 1}]$

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February 2, 2008

Abstract

Hartwig, Larsson and the second author in [J. Algebra, 295 (2006)] defined a bracket on σ-derivations of a commutative algebra. We show that this bracket preserves inner derivations, and based on this obtain some structural results on σ-derivations on Laurent polynomials in one variable.

Keywords: quasi-Lie algebras, σ-derivations, twisted bracket, q-deformed Witt algebras.
Mathematics Subject Classification 2000: 17A36; 17B40; 17B66; 17B68; 81R12; 16W55; 16S35

1 Introduction

In [15, 22, 23, 24] a new class of algebras called quasi-Lie algebras and its subclasses, quasi-hom-Lie algebras and hom-Lie algebras, have been introduced. An important characteristic feature of those algebras is that they obey some deformed or twisted versions of skew-symmetry and Jacobi identity with respect to some possibly deformed or twisted bilinear bracket multiplication. Quasi-Lie algebras include color Lie algebras, and in particular Lie algebras and Lie superalgebras, as well as various interesting quantum deformations of Lie algebras, in particular of the Heisenberg Lie algebra, oscillator algebras,

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sl_2 and other finite-dimensional Lie algebras as well as of infinite-dimensional Lie algebras of Witt and Virasoro type applied in physics within the string theory, vertex operator models, quantum scattering, lattice models and other contexts (see [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20] and references therein). Many of these quantum deformations of Lie algebras can be shown to play role of underlying algebraic objects for calculi of twisted, discretized or deformed derivations and difference type operators and thus in corresponding general non-commutive differential calculi.

In [15, Theorem 5], it was proved that under some general assumptions, when derivations are replaced by twisted derivations, vector fields become replaced by twisted vector fields closed under a natural twisted skew-symmetric bracket multiplication satisfying a twisted 6-term Jacobi identity generalizing the usual Lie algebras 3-term Jacobi identity that is recovered when no twisting is present (see Theorem 2.2.2). This theorem is shown to be instrumental for construction of various examples and classes of quasi-Lie algebras. Both known and new one-parameter and multi-parameter deformations of Witt and Virasoro algebras and other Lie and color Lie algebras has been constructed within this framework in [15, 22, 23, 24].

In this article, we gain further insight in the particular class of quasi Lie algebra deformations of the Witt algebra, introduced in [15] via the general twisted bracket construction, and associated with twisted discretization of derivations generalizing the Jackson q-derivatives to the case of twistings by general endomorphisms of Laurent polynomials. In Section 2 we present necessary definitions, facts and constructions on σ-derivations that are central for this article. In Proposition 2.3.1 we observe that inner derivations are stable under the bracket defined in [15]. In Section 3, we present a characterization of the set of inner derivations for UFD (Proposition 3.1), and also general inclusions concerning sets of inner derivations and image and pre-image subsets with respect to the twisted bracket (Proposition 3.8). In Section 4 we develop the preceding framework for a particular important UFD, the algebra $\mathcal{A} = \mathbb{C}[t^{\pm 1}]$ of Laurent polynomials in one variable. For this specialization more deep and precise results can be obtained. In this case, $\mathcal{D}_\sigma(\mathcal{A})$ with the twisted bracket is the deformation of the Witt algebra within the class of quasi-hom-Lie algebras in the sense of [15]. In Theorem 4.2.1 we show that the space of σ-derivations can be decomposed into a direct sum of the space of inner σ-derivations and a dependent on σ finite number of one-dimensional subspaces. In Theorem 4.3.3, we show that for arbitrary σ the $\mathbb{Z}$-gradation of this non-linearly q-deformed Witt algebra with coeffi-

cients in $\mathbb{C}$ becomes a $\mathbb{Z}/d\mathbb{Z}$-gradation with coefficients in $\mathbb{C}[T^{\pm 1}]$ for some element T of A. The usual q-deformed Witt algebra associated to ordinary Jackson q-derivative, the automorphism case, corresponds to $d = 0$, when all σ-derivations are inner. In Subsection 4.4, we provide more detailed description of what relations for the bracket in the non-linearly q-deformed Witt algebra become modulo inner σ-derivations. Finally, in Subsection 4.5, we describe stabilizer-like subsets in detail for the non-linearly q-deformed Witt algebra.

Throughout this article, $\mathcal{A}$ will denote an associative and unital algebra over the field $\mathbb{C}$ of complex numbers. The algebra $\mathcal{A}$ will often be assumed to be commutative, but we will sometimes mention more general results concerning non-commutative algebras, so we may precise our assumptions on $\mathcal{A}$ every time.

2 Some general facts on σ-derivations

2.1 Definitions

We recall here some basic definitions and facts concerning σ-derivations. On this subject and more generally on Ore extensions one may see the reference book [29], or section 1.7 in [21].

Definition 2.1.1 *Let $\mathcal{A}$ be an algebra, and σ an endomorphism of $\mathcal{A}$. A σ-derivation is a linear map D satisfying $D(ab) = \sigma(a)D(b) + D(a)b$ for all $a, b \in \mathcal{A}$. We denote the set of all σ-derivations by $\mathcal{D}_\sigma(\mathcal{A})$.*

Example. It is easy to check that for any $p \in \mathcal{A}$, the $\mathbb{C}$-linear map Δ_p defined by $\Delta_p(a) = pa - \sigma(a)p$ for all $a \in \mathcal{A}$ is a σ-derivation of $\mathcal{A}$. Note that if $\mathcal{A}$ is commutative, then we have $\Delta_p = p(\mathrm{id} - \sigma)$.

Definition 2.1.2 *The map Δ_p defined above is called the* inner *σ-derivation associated to p. The set of all inner derivations of $\mathcal{A}$ will be denoted $\mathcal{I}nn_\sigma(\mathcal{A})$.*

For any map $\tau : \mathcal{A} \to \mathcal{A}$ denote $\mathrm{Ann}(\tau) = \{a \in \mathcal{A} \mid a\tau(b) = 0 \ \forall\ b \in \mathcal{A}\}$, the left annihilator ideal of τ. In particular if $\mathcal{A}$ is commutative then this is a two sided ideal, and also $\Delta_p = \Delta_q \iff (p - q) \in \mathrm{Ann}(\mathrm{id} - \sigma)$.

The σ-derivations play a crucial role in the definition of Ore extensions, that we recall here.

Definition 2.1.3 *Let $\mathcal{A}$ be an algebra, σ an endomorphism of $\mathcal{A}$, and Δ a σ-derivation. Then the Ore extension $R = \mathcal{A}[X; \sigma, \Delta]$ is the algebra such that:*

- *$\mathcal{A}$ is a sub-algebra of R;*
- *R is a free $\mathcal{A}$-module with basis $\{X^n,\ n \in \mathbb{N}\}$;*
- *the multiplication is defined in R by the rule $Xa = \sigma(a)X + \Delta(a)$ for all $a \in \mathcal{A}$.*

The following facts can be easily checked; they appear in [14, Lemmas 1.5 and 2.4.].

Lemma 2.1.4 *Let σ be an endomorphism of an algebra $\mathcal{A}$ and Δ_p be an inner derivation corresponding to an element $p \in A$.*

1. *Set $p \in \mathcal{A}$, and $\Delta_p \in \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$. Then the identity map on $\mathcal{A}$ extends to an isomorphism τ between the Ore extensions $\mathcal{A}[X; \sigma, \Delta_p]$ and $\mathcal{A}[Y; \sigma]$, defined by $\tau(X) = Y + p$ sending X to $Y + p$.*
2. *Assume $\mathcal{A}$ is commutative. Then for all $\Delta \in \mathcal{D}_\sigma(\mathcal{A})$, and for all $a, b \in \mathcal{A}$ one has*
$$(a - \sigma(a))\Delta(b) = (b - \sigma(b))\Delta(a).$$

The first statement of this lemma provides one of the reasons why we get interested in σ-derivations up to inner in the following sections.

2.2 A bracket on σ-derivations

From now on $\mathcal{A}$ is supposed to be a *commutative* algebra. Then $\mathcal{D}_\sigma(\mathcal{A})$ becomes a left $\mathcal{A}$-module by $(a\Delta)(r) = a\Delta(r)$ for all $a, r \in \mathcal{A}$. Note that in the non-commutative case this operation makes $\mathcal{D}_\sigma(\mathcal{A})$ a left module only over the center of $\mathcal{A}$.

Now we fix a σ-derivation Δ, and consider the cyclic left $\mathcal{A}$-submodule of $\mathcal{D}_\sigma(\mathcal{A})$ generated by Δ. The interest of considering cyclic sub-modules is reinforced by the following result proved by Hartwig, Larsson and the second author in [15], Theorem 2, in the case where $\mathcal{A}$ is a unique factorization domain.

Theorem 2.2.1 ([15]) *Let σ be an endomorphism of a unique factorization domain $\mathcal{A}$, and $\sigma \neq \mathrm{id}$. Then $\mathcal{D}_\sigma(\mathcal{A})$ is free of rank one as an $\mathcal{A}$-module with generator*

$$\Delta = \frac{\mathrm{id} - \sigma}{g}, \text{with} \quad g = \gcd((\mathrm{id} - \sigma)(\mathcal{A})).$$

They also define in [15], Theorem 5, a bracket on this cyclic sub-module and prove the following results.

Theorem 2.2.2 ([15]) *Let σ be an endomorphism of a commutative algebra $\mathcal{A}$, and $\sigma \neq \mathrm{id}$. Let $\Delta \in \mathcal{D}_\sigma(\mathcal{A})$ be a σ-derivation such that :*

- $\sigma(\mathrm{Ann}(\Delta)) \subseteq \mathrm{Ann}(\Delta)$;
- $\exists\ \delta \in \mathcal{A}$ *such that* $\Delta \circ \sigma = \delta\sigma \circ \Delta$.

then the map

$$[\cdot,\cdot]_\sigma : \mathcal{A}\Delta \times \mathcal{A}\Delta \to \mathcal{A}\Delta$$

defined by setting

$$[a\Delta, b\Delta]_\sigma = (\sigma(a)\Delta) \circ (b\Delta) - (\sigma(b)\Delta) \circ (a\Delta), \quad \textit{for } a, b \in \mathcal{A}, \tag{1}$$

where $\circ$ denotes composition of functions, is a well-defined $\mathbb{C}$-algebra product on the $\mathbb{C}$-linear space $\mathcal{A}\Delta$, satisfying the following identities for $a, b, c \in \mathcal{A}$:

$$[a\Delta, b\Delta]_\sigma = (\sigma(a)\Delta(b) - \sigma(b)\Delta(a))\Delta, \tag{2}$$

$$[a\Delta, b\Delta]_\sigma = -[b\Delta, a\Delta]_\sigma. \tag{3}$$

In addition,

$$\begin{aligned} &[\sigma(a)\Delta, [b\Delta, c\Delta]_\sigma]_\sigma + \delta[a\Delta, [b\Delta, c\Delta]_\sigma]_\sigma + \\ &+[\sigma(b)\Delta, [c\Delta, a\Delta]_\sigma]_\sigma + \delta[b\Delta, [c\Delta, a\Delta]_\sigma]_\sigma + \\ &+[\sigma(c)\Delta, [a\Delta, b\Delta]_\sigma]_\sigma + \delta[c\Delta, [a\Delta, b\Delta]_\sigma]_\sigma = 0. \end{aligned} \tag{4}$$

Remark 2.2.3

1. The second condition, under the extra assumption that $\delta = q \in \mathbb{C}^*$, is the definition of q-skew derivations given in [14]. These particular σ-derivations play an important role in quantum groups, see for instance [2] or the reference book [1] and references therein. Note that in this case Formula (4) can be written as a 3-term Jacobi-like identity.

2. The identity (2) is just a formula expressing the product defined in (1) as an element of $\mathcal{A}\Delta$. Identities (3) and (4) are expressing, respectively, skew-symmetry and a generalized $((\sigma, \delta)$-twisted) Jacobi identity for the product defined by (1).

2.3 Inner σ-derivations

We recall from Definition 2.1.2 that a σ-derivation $\widetilde{\Delta}$ is inner if and only if there exists an element $p \in \mathcal{A}$ such that $\widetilde{\Delta}(a) = pa - \sigma(a)p$ for all $a \in \mathcal{A}$. Because $\mathcal{A}$ is commutative, it is easy to see that if Δ is inner, then $a\Delta$ is inner for all $a \in \mathcal{A}$, so that $\mathcal{A}\Delta \subseteq \mathcal{I}nn_\sigma(\mathcal{A})$. So we mainly get interested in the case where Δ itself is not inner.

Now we prove that inner derivations are stable under the bracket $[.,.]_\sigma$.

Proposition 2.3.1 *Let σ be an endomorphism of a commutative algebra $\mathcal{A}$, and $\Delta \in \mathcal{D}_\sigma(\mathcal{A})$. Set $a, b \in \mathcal{A}$ such that $a\Delta = p(\mathrm{id} - \sigma)$ and $b\Delta = q(\mathrm{id} - \sigma)$ are inner. Then $[a\Delta, b\Delta]_\sigma$ is inner. More precisely, we have: $[a\Delta, b\Delta]_\sigma = c(\mathrm{id} - \sigma)$, with $c = \Delta(b)p - \Delta(a)q$, and so $[a\Delta, b\Delta]_\sigma = (\Delta(b)a - \Delta(a)b)\Delta$.*

Proof. For all $r \in \mathcal{A}$ one has:

$$[a\Delta, b\Delta]_\sigma(r) = (\sigma(a)\Delta(b) - \sigma(b)\Delta(a))\Delta(r) = \sigma(a)\Delta(b)\Delta(r) - \sigma(b)\Delta(a)\Delta(r).$$

From Lemma 2.1.4 we have $(r - \sigma(r))\Delta(a) = (a - \sigma(a))\Delta(r)$. Using the fact that $a\Delta(r) = p(r - \sigma(r))$ and $\mathcal{A}$ is commutative we get then $\sigma(a)\Delta(r) = (r - \sigma(r))(p - \Delta(a))$. In the same way one proves that $\sigma(b)\Delta(r) = (r - \sigma(r))(q - \Delta(b))$. So:

$$\begin{aligned} [a\Delta, b\Delta]_\sigma(r) &= \Delta(b)(r - \sigma(r))(p - \Delta(a)) - \Delta(a)(r - \sigma(r))(q - \Delta(b)) \\ &= (\Delta(b)p - \Delta(a)q)(r - \sigma(r)). \end{aligned}$$

□

Remark 2.3.2
1. In the UFD case, since $a = gp$ and $b = gq$, then one gets $c = \sigma(g)(\Delta(q)p - \Delta(p)q)$.
2. $\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$ appears as a sub-algebra of $(\mathcal{D}_\sigma(\mathcal{A}), [.,.]_\sigma)$. It is the sub-algebra considered in paragraph 3.2.1 of [15].
3. In the non-commutative case, under the assumption that Δ sends the center of $\mathcal{A}$ to itself, we can prove in the same way for $a, b \in Z(\mathcal{A})$ and with the notations of Definition 2.1.2, that if $a\Delta = \Delta_p$ and $b\Delta = \Delta_q$ then $[a\Delta, b\Delta]_\sigma = \Delta_t$, with $t = \Delta(b)p - \Delta(a)q$.

3 The UFD case.

We present in this section some general statements concerning σ-derivations in the UFD case. In the next section we will give some more precise and deep results in the particular case $\mathcal{A} = \mathbb{C}[t^{\pm 1}]$. One can first precise Theorem 2.2.1 in the following way.

Proposition 3.1 *Let σ be an endomorphism of $\mathcal{A}$ a unique factorization domain, $\sigma \neq \mathrm{id}$. Set $g = \gcd((\mathrm{id} - \sigma)(\mathcal{A}))$. Then*

1. *$\mathcal{D}_\sigma(\mathcal{A}) = \mathcal{A}\Delta$, with $\Delta = \frac{\mathrm{id}-\sigma}{g}$;*
2. *the σ-derivation $a\Delta$ is inner if and only if g divides a. In other words, $\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A}) = g\mathcal{A}\Delta$.*

Proof. The first point is just Theorem 2.2.1. Then any σ-derivation can be written $\widetilde{\Delta} = a\Delta$, with $a \in \mathcal{A}$. Obviously if $a = bg$ then $a\Delta = b(\mathrm{id} - \sigma)$ is inner.

Conversely, assume that $a\Delta$ is inner. Then there is an element $b \in \mathcal{A}$ such that $a\Delta(r) = b(r - \sigma(r))$ for all $r \in \mathcal{A}$. Multiplying by g one obtains $ag\Delta(r) = b(r - \sigma(r))$, i.e. $a(r - \sigma(r)) = bg(r - \sigma(r))$ by definition of Δ. Now choose r such that $r - \sigma(r) \neq 0$, and conclude using the fact that $\mathcal{A}$ is a domain. $\square$

In particular, Δ itself is inner if and only if g is a unit.

In order to use the bracket $[.,.]_\sigma$ to understand what is "between" $\mathcal{D}_\sigma(\mathcal{A})$ and $\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$, we define now some sub-spaces of $\mathcal{D}_\sigma(\mathcal{A})$, which will be more precisely described in the next section for $\mathcal{A} = \mathbb{C}[t^{\pm 1}]$. Recall that $\mathcal{D}_\sigma(\mathcal{A}) = \mathcal{A}\Delta$ and $\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A}) = g\mathcal{A}\Delta$.

Definition 3.2 $S^1 = \mathrm{Span}_{\mathbb{C}}[\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A}), \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})]_\sigma$.

Remark 3.3
1. S^1 would be the usual derived Lie algebra of $\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$ for $\sigma = \mathrm{id}$.
2. It follows from Proposition 2.3.1 that one always have $S^1 \subseteq \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$.

Lemma 3.4 *$S^1 \subseteq \sigma(g)g\mathcal{A}\Delta$.*

Proof. This inclusion relies on the remark following Proposition 2.3.1. We are in the UFD case, and if $a, b \in \mathcal{A}$ such that $a\Delta, b\Delta \in \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$, then $a = gp$ and $b = gq$ for some $p, q \in \mathcal{A}$, and $[a\Delta, b\Delta]_\sigma = \sigma(g)(\Delta(q)p - \Delta(p)q)g\Delta$. So $[\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A}), \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})]_\sigma \subseteq \sigma(g)g\mathcal{A}\Delta$. $\square$

The following corollary is a direct consequence of the preceding Lemma and part 2 of Proposition 3.1.

Corollary 3.5 *If $\sigma(g)$ is not a unit in $\mathcal{A}$, then $S^1 \subset \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$.*

Now we define two subspaces of $\mathcal{D}_\sigma(\mathcal{A})$. Note that these definitions can be given for any algebra, no matter if it is a UFD or not.

Definition 3.6 $\widetilde{S}_1 = \{\widetilde{\Delta} \in \mathcal{D}_\sigma(\mathcal{A}) |\ [\widetilde{\Delta}, S^1] \subseteq \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})\}$.

Because $S^1 \subseteq \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$ it follows from Proposition 2.3.1 that $\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A}) \subseteq \widetilde{S}_1$. As we will see in the case of q-deformed Witt algebras $\widetilde{S}_1$ is the whole space of σ-derivations, so we define now a "smaller" space.

Definition 3.7 $S_1 = \{\widetilde{\Delta} \in \mathcal{D}_\sigma(\mathcal{A}) \mid [\widetilde{\Delta}; S^1]_\sigma \subseteq S^1\}$.

Because $S^1 \subseteq \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$ it is clear that $S_1 \subseteq \widetilde{S}_1$. This is the stabilizer of S^1 in $\mathcal{D}_\sigma(\mathcal{A})$ with respect to $[.,.]_\sigma$.

We can summerize the inclusions as follows.

Proposition 3.8 *Let $\mathcal{A}$ be a UFD. Then $S^1 \subseteq \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A}) \subseteq S_1 \subseteq \widetilde{S}_1$.*

4 Non-linearly q-deformed Witt algebras

We develop now the preceding framework for a particular algebra $\mathcal{A}$, namely $\mathbb{C}[t^{\pm 1}]$, in order to obtain some more deep and precise results. We study the case $\mathcal{A} = \mathbb{C}[t^{\pm 1}]$, so $\mathcal{D}_\sigma(\mathcal{A})$ with the twisted bracket is the deformation of the Witt algebra in the sense of [15]. The deformations of the Witt algebra are of importance in mathematical physics (see [15] and references therein, and the introduction of the present work). Note that as $\mathbb{C}[t^{\pm 1}]$ is a UFD Proposition 3.1 and Proposition 3.8 apply.

Most of the articles on σ-derivations of $\mathbb{C}[t^{\pm 1}]$ are concerned with the case where σ is an automorphism (see for instance [20], [30]). We don't assume here that σ is an automorphism, so it involves some power s of t, which as we will see plays a crucial role in the study of $\mathcal{D}_\sigma(\mathcal{A})$.

4.1 Some notations

Degree, valuation. For a Laurent polynomial $f(t) = \sum_{n=n_0}^{n_1} \alpha_n t^n$ with $\alpha_n \in \mathbb{C}$, $\alpha_{n_0} \neq 0$, $\alpha_{n_1} \neq 0$ we denote $\nu(f) = n_0$ the valuation of f and $\deg(f) = n_1$ its degree.

The endomorphism σ. Because an endomorphism of algebra sends units on units, the image of t by σ must be a monomial. So denote $\sigma(t) = qt^s$, with $q \in \mathbb{C}^*$ and $s \in \mathbb{Z}$. Note that σ is injective if and only if $s \neq 0$, and surjective if and only if $s = 1$ or $s = -1$.

Generators of $\mathcal{D}_\sigma(\mathcal{A})$. If $\sigma \neq \mathrm{id}$ then by Theorem 2.2.1 one has $\mathcal{D}_\sigma(\mathcal{A}) = \mathcal{A}\Delta$ with $\Delta(f) = (f - \sigma(f))/g$, $g = \gcd((\mathrm{id} - \sigma)(\mathcal{A}))$. Then one can check (see [15], example 3.2) that $g = \alpha^{-1} t^{k-1}(t - qt^s)$ with $\alpha \in \mathbb{C}^*$ and $k \in \mathbb{Z}$. Since g is defined up to a unit, then α and k are arbitrary. If $s \geq 1$ then choose $k = 0$ and $\alpha = 1$, so that $g(t) = 1 - qt^{s-1}$. If $s \leq 0$ then choose $k = -s+1$ and $\alpha = -q$, so that $g(t) = -q^{-1}(t^{1-s} - q) = 1 - q^{-1}t^{1-s}$. In both cases $g(t)$ is a (non-Laurent) polynomial of degree $|s-1|$ such that $g(0) = 1$. So we can assume without loss of generality that

$$g(t) = 1 - \lambda t^d,$$

with $[\lambda = q, d = s-1]$ if $s \geq 1$ and $[\lambda = q^{-1}, d = 1-s]$ if $s < 1$. So for $s \neq 1$ one has $d \geq 1$ and $\lambda \neq 0$. Note that with our conventions, for $\sigma = \mathrm{Id}$ (i.e. $s = 1 = q$), one gets $g = 0$. We consider the following linear basis of $\mathcal{D}_\sigma(\mathcal{A})$ (see Example 3.2 in [15]): $d_n = -t^n\Delta$ for all $\mathbb{Z}$.

The monomial $T = qt^{s-1}$ will play a crucial rule in the following. Note that $g = 1 - T$ if $s \geq 1$, and $g = 1 - T^{-1}$ if $s < 1$. Note also that $\sigma(T) = T^s$, and that T "acts" on $\mathcal{D}_\sigma(\mathcal{A})$ in the following way: $Td_n = qd_{n+s-1}$. At last, since $\Delta(t) \neq 0$ and $\mathcal{A}$ is a domain, we have $\mathrm{Ann}(\Delta) = \{0\}$.

4.2 Decomposition of $\mathcal{D}_\sigma(\mathbb{C}[t^{\pm 1}])$

If $s = 1$ then two cases may occur. First if $q = 1$ then σ is the identity map, so one gets the usual Witt algebra, and we won't consider this case here. Second case: if $q \neq 1$ then $g = 1 - q$ is a unit, and Proposition 3.1 implies that all σ-derivations are inner, and $\mathcal{D}_\sigma(\mathcal{A}) = \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A}) = S^1 = \widetilde{S}_1 = S_1$. As we get interested in the study of what happens "between" $\mathcal{D}_\sigma(\mathcal{A})$ and $\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$, we shall assume that $s \neq 1$. Note that then g is not a unit in $\mathcal{A}$, so thanks to Corollary 3.5 we have $S^1 \subset \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$.

The vector space $\mathcal{D}_\sigma(\mathcal{A})$ is made a non-associative algebra thanks to the bracket $[.,.]_\sigma$ defined in Theorem 2.2.2. Since g is not a unit, the set $\mathcal{I}nn_\sigma(\mathcal{A})$ of inner σ-derivations is a proper subalgebra of $(\mathcal{D}_\sigma(\mathcal{A}), [.,.]_\sigma)$. Moreover, thanks to Proposition 3.1 we know that a σ-derivation $f\Delta$ is inner if and only if g divides f, that is $\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A}) = g\mathcal{A}\Delta$. This leads to the following result.

Theorem 4.2.1 *Assume that $s \neq 1$. Then with the notations above*

$$\mathcal{D}_\sigma(\mathcal{A}) = \mathbb{C}d_0 \oplus \mathbb{C}d_1 \ldots \oplus \mathbb{C}d_{d-1} \oplus \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A}).$$

Proof. Note that for any $n \in \mathbb{N}$ one has $\mathbb{C}d_n = \mathbb{C}t^n\Delta$. We first show that $\mathcal{D}_\sigma(\mathcal{A}) = \mathbb{C}\Delta + \mathbb{C}t\Delta \ldots + \mathbb{C}t^{d-1}\Delta + \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$. Take any $\widetilde{\Delta} = f(t)\Delta \in \mathcal{D}_\sigma(\mathcal{A})$, with $f(t) = \sum_{n=n_0}^{n_1} \alpha_n t^n$, with $\nu(f) = n_0$ and $\deg(f) = n_1$. Up to an inner derivation we can always assume that $\nu(f) \geq 0$. If not, consider $f_1 = f - \alpha_{n_0} t^{n_0} g$: we have $f\Delta = f_1\Delta + \alpha_{n_0} t^{n_0} g\Delta$, $\alpha_{n_0} t^{n_0} g\Delta \in \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$ and $\nu(f_1) > \nu(f)$. If $\nu(f_1) \geq 0$ we are done, else we repeat this operation with f_1. Then after at most $\nu(f)$ iterations we have a polynomial $\tilde{f}$ such that $\nu(\tilde{f}) \geq 0$ and g divides $f - \tilde{f}$, that is $(f\Delta - \tilde{f}\Delta) \in \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$. So assume that $f \in \mathbb{C}[t]$. Now we can make the usual Euclidian division of f by g in $\mathbb{C}[t]$, and we obtain $f = q(t)g(t) + r(t)$, with $\deg(r) < \deg(g) = d$. Since $\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A}) = g\mathcal{A}\Delta$ we are done.

Now we prove that this sum is a direct sum. Set $\alpha_0, \ldots, \alpha_{d-1}$ in $\mathbb{C}$ and $f \in \mathbb{C}[t^{\pm 1}]$ such that $\sum_{i=0}^{d-1} \alpha_i t^i \Delta + fg\Delta = 0$. Since $\mathrm{Ann}(\Delta) = \{0\}$ this implies $\sum_{i=0}^{d-1} \alpha_i t^i + fg = 0$. First we prove that f must be a non-Laurent polynomial, that is $\nu(f) \geq 0$. Assume on the contrary that $\nu(f) = n_0 < 0$, and f has lowest degree term $\beta_{n_0} t^{n_0} \neq 0$. Then because $g = 1 - \lambda t^d$ with $d > 0$, the term of lowest degree of $\sum_{i=0}^{d-1} \alpha_i t^i + fg$ is $\beta_{n_0} t^{n_0}$, a contradiction since $\sum_{i=0}^{d-1} \alpha_i t^i + fg = 0$.

Now the latest equality is nothing else but the Euclidian division of the 0 polynomial by g in $\mathbb{C}[t]$. By uniqueness we have $\alpha_i = 0$ for all i and $f = 0$. $\square$

In Subsection 4.4 we will give a description of the bracket in $\mathcal{D}_\sigma(\mathcal{A})$ in terms of this decomposition, thanks to the brackets computed in [15]. But first we re-interpret these in term of the element $T = qt^{s-1}$ defined at the beginning of this section, and show that the algebra $\mathcal{D}_\sigma(\mathcal{A})$ is graded by a finite cyclic group.

4.3 Grading of $\mathcal{D}_\sigma(\mathcal{A})$

We recall first the following from [15, Theorem 8].

Theorem 4.3.1 *Let $\mathcal{A} = \mathbb{C}[t^{\pm 1}]$, and equip the $\mathbb{C}$-linear space $\mathcal{D}_\sigma(\mathcal{A}) = \bigoplus_{n\in\mathbb{Z}} \mathbb{C} d_n$ with the bracket product*

$$[\,\cdot,\cdot\,]_\sigma : \mathcal{D}_\sigma(\mathcal{A}) \times \mathcal{D}_\sigma(\mathcal{A}) \longrightarrow \mathcal{D}_\sigma(\mathcal{A})$$

defined on generators by (1) as

$$[d_n, d_m]_\sigma = q^n d_{ns} d_m - q^m d_{ms} d_n. \tag{5}$$

This bracket satisfies defining commutation relations

$$\begin{aligned}
[d_n, d_m]_\sigma &= \alpha \operatorname{sign}(n-m) \sum_{l=\min(n,m)}^{\max(n,m)-1} q^{n+m-1-l} d_{s(n+m-1)-(k-1)-l(s-1)} \\
&\quad \text{for } n,m \geq 0; \\
[d_n, d_m]_\sigma &= \alpha \Big(\sum_{l=0}^{-m-1} q^{n+m+l} d_{(m+l)(s-1)+ns+m-k} \\
&\qquad\qquad + \sum_{l=0}^{n-1} q^{m+l} d_{(s-1)l+n+ms-k} \Big) \\
&\quad \text{for } n \geq 0, m < 0; \\
[d_n, d_m]_\sigma &= -\alpha \Big(\sum_{l_1=0}^{m-1} q^{n+l_1} d_{(s-1)l_1+m+ns-k} \\
&\qquad\qquad + \sum_{l_2=0}^{-n-1} q^{m+n+l_2} d_{(n+l_2)(s-1)+n+ms-k} \Big) \\
&\quad \text{for } m \geq 0, n < 0; \\
[d_n, d_m]_\sigma &= \alpha \operatorname{sign}(n-m) \sum_{l=\min(-n,-m)}^{\max(-n,-m)-1} q^{n+m+l} d_{(m+n)s+(s-1)l-k} \\
&\quad \text{for } n,m < 0.
\end{aligned}$$

Furthermore, this bracket satisfies skew-symmetry $[d_n, d_m]_\sigma = -[d_m, d_n]_\sigma$ and a twisted Jacobi identity.

Note that for $s = 1$ formula (5) gives the usual formula for the q-Witt algebra, or the Witt algebra if $q = 1$, that is

$$[d_n, d_m]_q = q^n d_n d_m - q^m d_m d_n.$$

Now for arbitrary $s \in \mathbb{Z}$, recall that $T = qt^{s-1}$, and $\sigma(T) = T^s$. Assume σ is not the identity map (so that $T \neq 1$), and for all $n \in \mathbb{Z}$ note the T-integer $\{n\}_T = \frac{T^n-1}{T-1}$. This is just a geometric sum, more precisely one has $\{0\}_T = 0$, $\{n\}_T = \sum_{k=0}^{n-1} T^k$ for $n > 0$ and $\{n\}_T = -\sum_{k=n}^{-1} T^k$ for $n < 0$. Thanks to these notations we shall rewrite the preceding formulas in the following way.

Proposition 4.3.2 *For all $n < m \in \mathbb{Z}$ one has*

$$[d_n; d_m]_\sigma = T^n d_n d_m - T^m d_m d_n;$$

$$[d_n; d_m]_\sigma = -(T^n \sum_{k=0}^{m-n-1} T^k) d_{n+m} = (\{n\}_T - \{m\}_T) d_{n+m}. \tag{6}$$

Proof. The first equation comes directly from (5) and the definitions of T and of the d_n's. For the second one, we begin with the case $0 \leq n < m$:

$$[d_n, d_m]_\sigma = \sum_{l=n}^{m-1} q^{n+m-1-l} t^{s(n+m-1)+1-l(s-1)} \Delta;$$

which becomes after reindexing the summation with $k = m - 1 - l$:

$$\begin{aligned}
[t^n\Delta, t^m\Delta]_\sigma &= \sum_{k=0}^{m-n-1} q^{n+k} t^{s(n+m-1)+1-(s-1)(m-1-k)} \Delta \\
&= \sum_{k=0}^{m-n-1} q^{n+k} t^{sn+m+k(s-1)} \Delta \\
&= (q^n t^{sn+m} \sum_{k=0}^{m-n-1} q^k t^{(s-1)k}) \Delta \\
&= t^{n+m} (qt^{s-1})^n \sum_{k=0}^{m-n-1} (qt^{s-1})^k \Delta.
\end{aligned}$$

The two other cases are treated in the same way, thanks to the following formulas : if $n < 0 \leq m$ then

$$\begin{aligned}
[t^n\Delta, t^m\Delta]_\sigma &= \sum_{l=0}^{m-1} q^{n+l} t^{l(s-1)+m+ns} \Delta + \sum_{l=0}^{-n-1} q^{m+n+l} t^{(n+l)(s-1)+n+ms} \Delta \\
&= t^{n+m} (qt^{s-1})^n \sum_{l=0}^{m-1} q^l t^{l(s-1)} \Delta \\
&\qquad + \sum_{l=m}^{m-n-1} q^{n+l} t^{(n+l-m)(s-1)+n+ms} \Delta \\
&= t^{n+m} (qt^{s-1})^n \sum_{k=0}^{m-n-1} (qt^{s-1})^k \Delta;
\end{aligned}$$

and finally if $n < m < 0$ then

$$\begin{aligned}
[t^n\Delta, t^m\Delta]_\sigma &= \sum_{l=-m}^{-n-1} q^{n+m+l} t^{(m+n)s+(s-1)l}\Delta \\
&= \sum_{l=0}^{m-n-1} q^{n+l} t^{(m+n)s+(s-1)(l-m)}\Delta \\
&= t^{n+m}(qt^{s-1})^n \sum_{k=0}^{m-n-1} (qt^{s-1})^k \Delta.
\end{aligned}$$

□

Let us remark here that the second expression in formula (6) shows that these non-linearly deformed Witt algebras, constructed *a priori* in [15] by taking any endomorphism of $\mathbb{C}[t^{\pm 1}]$ instead of the automorphism $t \to qt$, really "look like" the q-Witt algebra. More precisely, if one takes for σ the automorphism $t \to qt$, then $s = 1$, so $d = 0$, and $T = q$. Then the T-integers are the usual q-integers, and (6) is the usual bracket of the q-Witt algebra, as defined for instance in [20], and leading for $q = 1$ to the classical Witt algebra.

We show now that for arbitrary σ the $\mathbb{Z}$-gradation of the q-Witt algebra with coefficients in $\mathbb{C}$ becomes a $\mathbb{Z}/d\mathbb{Z}$-gradation with coefficients in $\mathbb{C}[T^{\pm 1}]$. Once again this is relevant, because the usual q-Witt algebra corresponds to the case where $d = 0$.

Theorem 4.3.3 *Let σ be the endomorphism of $\mathcal{A} = \mathbb{C}[t^{\pm 1}]$ defined by $\sigma(t) = qt^s$, with $q \in \mathbb{C}^*$ and $s \in \mathbb{Z}$. Recall that $\mathcal{D}_\sigma(\mathcal{A})$ is the space of σ-derivations of $\mathcal{A}$, endowed with the bracket $[.,.]_\sigma$ defined in Theorem 2.2.2. Define $d = |s-1|$, and note $\mathbb{Z}_d = \mathbb{Z}/d\mathbb{Z}$, in particular for $s = 1$ one has $\mathbb{Z}/0 = \mathbb{Z}$. For any $k \in \mathbb{Z}$ note $\overline{k} = k + d\mathbb{Z} \in \mathbb{Z}_d$. The nonassociative algebra $(\mathcal{D}_\sigma(\mathcal{A}), [.,.]_\sigma)$ is $\mathbb{Z}_d$-graded: $\mathcal{D}_\sigma(\mathcal{A}) = \oplus_{\overline{k}\in\mathbb{Z}_d}\mathcal{D}_{\overline{k}}$, with $\mathcal{D}_{\overline{k}} = \mathbb{C}[T^{\pm 1}]d_k$ for any $k \in \overline{k}$.*

Proof. The case $s = 1$ is straightforward. So assume $s \neq 1$. Then note that $T = qt^d$ if $s > 1$, and $T = qt^{-d}$ if $s < 1$, with $d \geq 1$. So $\mathbb{C}[t^{\pm 1}] = \oplus_{i=0}^{d-1} t^i\mathbb{C}[T^{\pm 1}]$ as vector spaces. Now the direct sum in the Theorem follows from this and from the fact that $\mathrm{Ann}\Delta = \{0\}$. The grading results directly from formulas (6) and from the fact that $t^{n+m} = t^{n+m-d}q^{-1}T$ if $s > 1$, and $t^{n+m} = t^{n+m-d}qT^{-1}$ if $s < 1$. □

4.4 The bracket "modulo inner derivations"

Motivated by the previous results we are interested next in obtaining more detailed description of what relations for the bracket in [15, Theorem 8] (Theorem 4.3.1) become modulo inner σ-derivations.

Notation. We will use the following notation for congruence of two σ-derivations modulo inner σ-derivations: $\forall\ \widetilde{\Delta}, \widehat{\Delta} \in \mathcal{D}_\sigma(\mathcal{A})$, the expression $\widetilde{\Delta} \equiv \widehat{\Delta}$ means that $\widetilde{\Delta} - \widehat{\Delta} \in \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$.

Since $[\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A}), \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})]_\sigma \subseteq \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$, and the bracket is skew-symmetric, we shall only compute modulo $\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$ the brackets of 2 types:

- $[d_n, d_m]_\sigma$, with $0 \leq n < m \leq d-1$;
- $[d_n, gd_m]_\sigma$, with $m \in \mathbb{Z}$ and $0 \leq n \leq d-1$.

Note that since $T = qt^{s-1}$ is a unit in $\mathcal{A}$, and for $s < 1$ we have $1 - T = -T(1-T^{-1}) = -Tg$, so for all $s \in \mathbb{Z}$ we can write $\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A}) = (1-T)\mathcal{D}_\sigma(\mathcal{A})$.

Lemma 4.4.1 *For all $n \in \mathbb{N},\ m \in \mathbb{Z}$, one has*

1. $[d_n, d_m]_\sigma \equiv (n-m)d_{n+m}$;
2. *if* $s \geq 1$ *then* $d_m \equiv q^{-1}d_{m-d}$;
   if $s < 1$ *then* $d_m \equiv qd_{m-d}$.

Proof. 1. By Proposition 4.3.2 we know that:
$[d_n; d_m]_\sigma = -(T^n \sum_{k=0}^{m-n-1} T^k)d_{n+m}$. Since $\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A}) = (1-T)\mathcal{A}\Delta$, we must compute the remainder of the Euclidian division of the polynomial in T appearing in the right-hand side by $1-T$. Straightforward calculations show that for all $p \in \mathbb{Z},\ r \in \mathbb{N},\ r \geq 1$, we have in $\mathbb{C}[X^{\pm 1}]$

- $\sum\limits_{k=0}^{r} X^k = r + 1 - (1-X)(X^{r-1} + 2X^{r-2} + \ldots + r)$;
- if $p \geq 1$ then $X^p \sum\limits_{k=0}^{r} X^k = r + 1 - (1-X)(X^{p+r-1} + 2X^{p+r-2} + \ldots + rX^p + (r+1)(X^{p-1} + \ldots + 1))$;
- if $p \leq -1$ then $X^p \sum\limits_{k=0}^{r} X^k = r + 1 + (1-X)((r+1)(X^p + \ldots + X^{-1}) - X^{p+r-1} + 2X^{P+r-2} + \ldots + rX^p)$.

It results from these computations that the remainder is $(m-n)$.

2. Follows directly from the definition of g. $\square$

Proposition 4.4.2 *For all $0 \le n < m < d$ one has:*

1. *If $n+m < d$ then $[d_n, d_m]_\sigma \equiv (n-m)d_{m+n}$.*
   In particular for $n = 0$ one gets $[d_0, d_m]_\sigma = -md_m + \widetilde{\Delta}$, with $\widetilde{\Delta} \in \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$, and for $m = 1$ one has exactly $[d_0, d_1]_\sigma = -d_1$.

2. *If $n+m \ge d$ then*

   (a) $[d_n, d_m]_\sigma \equiv \frac{(n-m)}{q} d_{m+n-d}$ *if* $s \ge 1$*;*

   (b) $[d_n, d_m]_\sigma \equiv q(n-m)d_{m+n-d}$ *if* $s < 1$*.*

Proof. If $n+m < d$ we are done thanks to Lemma 4.4.1, part 1. One can easily check the case $n = 0, m = 1$. If $m+n \ge d$, we conclude using part 2 of Lemma 4.4.1. $\square$

Remark 4.4.3 Note that in the first formula, there is no more q appearing, just like in the classical Witt algebra. Unfortunately this does not induce such a formula on the quotient space $\mathcal{D}_\sigma(\mathcal{A})/\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$ because $\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$ is not an ideal for the $[.,.]_\sigma$ bracket, as results from the next proposition.

Proposition 4.4.4 *Assume $s \ne 1$. For all $0 \le n < d$, and all $m \in \mathbb{Z}$, set $p = \left[\frac{n+m}{d}\right]$ the integral part of $(n+m)/d$. Then $[d_n, gd_m]_\sigma \equiv -dq^{-\epsilon p} d_{m+n-pd}$, with $\epsilon = sign(s-1)$.*

Proof. We compute $[d_n, gd_m]_\sigma = -[t^n\Delta, t^m\Delta]_\sigma + q[t^n\Delta, t^{m+d}\Delta]_\sigma$, with $0 \le n < d$ and $m \in \mathbb{Z}$. Case $s > 1$: by Lemma 4.4.1 we get $[d_n, d_m]_\sigma \equiv (n-m)d_{n+m}$ and $[d_n, d_{m+d}]_\sigma \equiv -(m+d-n)t^{m+d+n}\Delta$. So

$$[d_n, gd_m]_\sigma \equiv (n-m)d_{n+m} - q(n-m-d)d_{n+m+d}$$
$$\equiv (n-m)(1-qt^d)d_{m+n} - qdd_{n+m+d} \equiv -qdd_{n+m+d} \equiv -dd_{n+m}\Delta.$$

So now it depends on the value of $n+m$. Recall that p is the only integer such that $dp \le n+m < d(p+1)$. So $0 \le n+m-dp < d$, and thanks to Lemma 4.4.1 part 2 we are done.

Case $s < 1$:

$$[t^n\Delta, t^m g\Delta]_\sigma = [t^n\Delta, t^m\Delta]_\sigma - q^{-1}[t^n\Delta, t^{m+d}\Delta]_\sigma \equiv$$
$$\equiv (n-m)t^{n+m}\Delta - q^{-1}(n-m-d)t^{n+m+d}\Delta$$
$$\equiv q^{-1}dt^{n+m+d}\Delta + (n-m)t^{n+m}(1-q^{-1}t^d)\Delta \equiv dt^{n+m}\Delta.$$

Once again we conclude thanks to part 2 of Lemma 4.4.1. $\square$

Remark 4.4.5 For $m = 0$ we get that $[d_n, g\Delta]_\sigma \equiv -dd_n$, from what we will deduce that $\{\widehat{\Delta} \in \mathcal{D}_\sigma(\mathcal{A}) \mid [\widehat{\Delta}, \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})]_\sigma \subseteq \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})\} = \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$.

4.5 The spaces S_1 and S^1

Recall that $\mathcal{D}_\sigma(\mathcal{A}) = \mathcal{A}\Delta$, $\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A}) = g\mathcal{A}\Delta$ and $[\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A}), \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})]_\sigma \subseteq \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$. The notations are the following: $\mathcal{A} = \mathbb{C}[t^{\pm 1}]$, $\sigma(t) = qt^s$, and $g = 1 - \lambda t^d$, with $d = s - 1$ and $\lambda = q$ if $s \geq 1$, and $d = 1 - s$ and $\lambda = q^{-1}$ if $s < 1$. We recall the following from Section 3.

Definition 4.5.1 *$S^1 = \mathrm{Span}_{\mathbb{C}}[\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A}), \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})]_\sigma$;*
$\widetilde{S}_1 = \{\widetilde{\Delta} \mid [\widetilde{\Delta}, S^1]_\sigma \subseteq \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})\}$;
$S_1 = \{\widetilde{\Delta} \in \mathcal{D}_\sigma(\mathcal{A}) \mid [\widetilde{\Delta}; S^1]_\sigma \subseteq S^1\}$.

Now we can describe these spaces in the case we are considering here.

Theorem 4.5.2 *1. If $s = 1$ then $\mathcal{D}_\sigma(\mathcal{A}) = \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A}) = S^1 = \widetilde{S}_1 = S_1$.*

2. If $s = 0$ then $\mathcal{D}_\sigma(\mathcal{A}) = \mathbb{C} \oplus \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A}) = \widetilde{S}_1 = S_1$, and $S^1 = 0$.

3. If $s \neq 0, 1$ then $S^1 \subset \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$ is a strict inclusion, and $\widetilde{S}_1 = \mathcal{D}_\sigma(\mathcal{A})$. Moreover, if $s \neq -1$ then $S_1 = \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$.

Proof. 1. This was already noted at the beginning of subsection 4.2.

2. The decomposition of $\mathcal{D}_\sigma(\mathcal{A})$ is Theorem 4.2.1. For the rest, just note that in this case $g = 1 - q^{-1}t$ and $\sigma(g) = 0$. So it follows from Lemma 3.4 that $S^1 = 0$. Then by definitions of these sets one gets $\widetilde{S}_1 = S_1 = \mathcal{D}_\sigma(\mathcal{A})$.

3. We begin with $\widetilde{S}_1$. We give the proof in the case $s > 1$, the case $s < 0$ is treated exactly in the same way, while changing the formulas. Note that $\sigma(g) = 1 - q^{d+1}t^{ds}$. We prove now that for all $\widetilde{\Delta} \in \mathcal{D}_\sigma(\mathcal{A})$ one has $[\widetilde{\Delta}, S^1] \subseteq \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$. It is enough to show that $\forall n, m$, with $0 \leq n < d$ and $m \in \mathbb{Z}$, one has $[t^n\Delta, \sigma(g)g\Delta]_\sigma \equiv 0$. But $[t^n\Delta, \sigma(g)g\Delta]_\sigma = [t^n\Delta, t^m g\Delta]_\sigma - q^{d+1}[t^n\Delta, t^{m+ds}g\Delta]_\sigma$. Thanks to Proposition 4.4.4 the right hand side is congruent up to inner derivations to

$$dq^{-p}t^{m+n-pd}\Delta - q^{d+1}dq^{-p-s}t^{m+ds+n-pd-sd}\Delta = 0,$$

where $p = [\frac{m+n}{d}]$.

Now we come to S_1 for $s \neq 0, 1, -1$. Because $S^1 \subset \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$ and by definition of S^1 it is clear that $S_1 \supseteq \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$. For the inverse inclusion, the following argument is valid for both $s > 1$ and $s < -1$, once we have noted that in both cases g divides $\sigma(g)$. We shall remark also that considering the Remark 2.3.2 for $q = t$ and $p = 1$ we get $\Delta(t)g\sigma(g)\Delta \in S^1$, with $\Delta(t) = t$ if $s > 1$ and $\Delta(t) = -qt^s$ for $s < -1$. Consider a σ-derivation $\tilde{P}(t)\Delta \in S_1$. According to Theorem 4.2.1 one should write $\tilde{P}(t) = \sum_{i=0}^{d-1} a_i t^i + g(t)R(t)$, so that $g(t)R(t)\Delta \in \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$. We must show that $P = \sum_{i=0}^{d-1} a_i t^i = 0$. We give details here for $s > 1$, the reader may check the case $s < -1$ in the same way. Because of Lemma 3.4 and the preceding remark we shall have $[P(t)\Delta, tg\sigma(g)\Delta]_\sigma \in g\sigma(g)\mathcal{A}\Delta$. Since this bracket is equal to $((\sigma(P)\Delta(tg\sigma(g)) - \Delta(P)\sigma(tg\sigma(g)))\Delta$ and $\mathrm{Ann}\Delta = 0$, we have $g\sigma(g)$ dividing $T = \sigma(P)\Delta(tg\sigma(g)) - \Delta(P)\sigma(tg\sigma(g))$. But $\sigma(tg\sigma(g)) = \sigma(t)\sigma(g)\sigma(\sigma(g))$ is a multiple of $g\sigma(g)$ because $\sigma(g)$ is a multiple of g. So $g\sigma(g)$ divides $\sigma(P)\Delta(tg\sigma(g))$. Now since Δ is a σ-derivation we have

$$\Delta(tg\sigma(g)) = qt^s\Delta(g\sigma(g)) + \Delta(t)g\sigma(g),$$

so $g\sigma(g)$ divides $\sigma(P)qt^s\Delta(g\sigma(g))$, i.e. it divides $\sigma(P)\Delta(g\sigma(g))$. Now, since $\Delta(g\sigma(g)) = \sigma(g)\Delta(\sigma(g)) + \Delta(g)\sigma(g)$, we get that g divides the polynomial $\sigma(P)(\Delta(\sigma(g)) + \Delta(g))$.

Now we prove that g and $Q = \Delta(\sigma(g)) + \Delta(g)$ are relatively prime, so by Gauss' Lemma g must divide $\sigma(P)$. By definition $\Delta = (\mathrm{id} - \sigma)/g$, so $\Delta(\sigma(g) + g) = (\sigma(g) - \sigma^2(g) + g - \sigma(g))/g = 1 - (\sigma^2(g)/g)$. Once again it is conveniant to notice that since $T = qt^d$, then $g = 1 - T$, $\sigma(g) = 1 - T^s$ and $\sigma^2(g) = 1 - T^{s^2}$. So $Q = -T\sum_0^{s^2-2} T^k$ and it is prime with g, because any root $t_0 \in \mathbb{C}$ of g satisfies $T(t_0) = 1$, so $Q(t_0) = 1 - s^2 \neq 0$ (by our hypothesis on s).

Finally we are reduced to the hypothesis that g divides $\sigma(P)$, with $P = \sum_{i=0}^{d-1} a_i t^i$. Then any root t_0 of g must be a root of P. Note that $g = 1 - \lambda t^d$ admits exactly d distinct roots (the d-roots of λ^{-1}) in $\mathbb{C}$. Let t_0 be one of these roots. Then $\sigma(P)(t_0) = \sum a_i(qt_0^s)^i = \sum a_i t_0^i$ since $t_0^d = q^{-1}$ and $d = s - 1$ if $s > 1$, and $t_0^d = q$ and $d = 1 - s$ if $s < -1$. So $P(t_0) = 0$, and P admits d distinct roots. Because P is of degree not higher than $d - 1$ this implies $P = 0$. □

Remark 4.5.3 1. Note that the computation of S_1 in the last case relies on the fact that $s^2 \neq 1$, and that is the reason why we could not treat the case $s = -1$;

2. Theorem 4.5.2 shows that for non-linearly q-deformed Witt algebra, "stabilizer-like" sets $S_1, \widetilde{S}_1$ while baring some information on relation between $\mathcal{D}_\sigma(\mathcal{A})$ and $\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$, do not provide a chain of subalgebras between $\mathcal{D}_\sigma(\mathcal{A})$ and $\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$.

Acknowledgements

This research was initiated during a course delivered by the second author at the Erasmus Intensive Program GAMAP: Geometric and Algebraic Methods of Physics and Applications in University of Antwerp in September 2005. Both authors wish to thank Prof. Fred Van Oystaeyen for kind hospitality and fruitful discussions. The first author thanks also the Centre for Mathematical Sciences at Lund University for hospitality during his one week stay there in February 2006.

The research was supported by the European Liegrits network, the British Engineering and Physical Sciences Research Council (EP/D034167/1), The Crafoord Foundation, The Swedish Foundation for Cooperation in Research and Higher Education (STINT), The Royal Physiographic Society in Lund and The Royal Swedish Academy of Sciences.

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